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ANISOTROPY OF ELASTIC PROPERTIES OF MATERIALS

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Papers dealing with the generalized Hooke's law for linearly elastic anisotropic media are reviewed. The papers considered are based on Kelvin's approach disclosing the structure of the generalized Hooke's law, which is determined by six eigenmoduli of elasticity and six orthogonal eigenstates.

Key words: *anisotropy, elasticity moduli, eigenmoduli and eigenstates, linearly elastic materials.*

Many natural materials, such as rocks, crystals, and biological tissues, and also materials used in advanced technologies, in particular, composites, are characterized by substantial anisotropy of their elasticity properties. In most cases, composite materials, as well as their components, are anisotropic materials. To create composite materials with elasticity properties necessary for engineering practice, one should know admissible limits of the components of the tensor of the elasticity moduli and the tensor of the compliance coefficients of anisotropic materials.

The constitutive equations of the linear theory of elasticity [1–5] in the Cartesian rectangular coordinate system (x_1, x_2, x_3) include the equations of motion

$$\sigma_{ij,j} - \rho \frac{\partial^2 u_i}{\partial t^2} + F_i = 0, \qquad i, j = 1, 2, 3, \tag{1}$$

the generalized Hooke's law

$$\sigma_{ij} = E_{ijkl}\varepsilon_{kl}, \qquad i, j, k, l = 1, 2, 3, \tag{2}$$

and the Cauchy formulas, which express strains via displacements:

$$2\varepsilon_{kl} = u_{l,k} + u_{k,l}. \tag{3}$$

In Eqs. (1)–(3), $\sigma_{ij} = \sigma_{ji}$ are the components of the symmetric stress tensor, $\varepsilon_{ij} = \varepsilon_{ji}$ are the components of the strain tensor, E_{ijkl} are the components of the fourth-rank tensor of the elasticity moduli, u_i are the components of the displacement vector, F_i are the components of the vector of bulk forces, ρ is the constant density of the material, and t is the time. The comma ahead of the subscript indicates differentiation with respect to the spatial coordinate marked by this subscript; repeated letters in the subscripts indicate summation over their admissible values. Relations (2) can be inverted: $\varepsilon_{ij} = S_{ijkl}\sigma_{kl}$ (S_{ijkl} are the components of the fourth-rank tensor of the compliance coefficients).

In the linear theory of elasticity, the specific strain energy for anisotropic materials is presented as [1, 2]

$$2\Phi = E_{ijkl}\varepsilon_{ij}\varepsilon_{kl} = S_{ijkl}\sigma_{ij}\sigma_{kl}. \tag{4}$$

The components E_{ijkl} possess the properties of symmetry [2, 5]:

$$E_{ijkl} = E_{jikl} = E_{klij}. \tag{5}$$

The constants S_{ijkl} also satisfy the conditions of symmetry (5) and are related to E_{ijkl} by the expressions

$$E_{ijkl}S_{klrs} = \delta_{ijrs} \equiv (\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr})/2, \qquad S_{ijkl}E_{klrs} = \delta_{ijrs},$$

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where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. The tensor δ_{ijrs} is a unit tensor in the space of symmetric tensors of the form (5).

The issues associated with the presentation of Hooke's law (2) in special bases and identification of the limits of variation of the constants E_{ijkl} compatible with the positive definiteness of the quadratic form (4) were considered in [1, 6–22]. The quadratic form (4) is reduced below to the canonical form, which allows understanding of the structure of the tensor E_{ijkl}.

Substituting relations (2) and (3) into Eq. (1), we obtain the equations of motion in displacements [2]

$$E^*_{ijkl} u_{j,kl} - \rho \frac{\partial^2 u_i}{\partial t^2} + F_i = 0, \tag{6}$$

where

$$E^*_{ijkl} = (E_{iklj} + E_{ilkj})/2. \tag{7}$$

The properties of the matrix E^*_{ijkl} were considered in [23, 24]. In solving particular problems, Eqs. (1)–(3) or (6) are supplemented by initial and boundary conditions. Equations (1) and (6) yield the static equations in the form

$$\sigma_{ij,j} + F_i = 0, \qquad E^*_{ijkl} u_{j,kl} + F_i = 0.$$

Let n_i and m_i ($i = 1, 2, 3$) be two orthogonal unit directions. Young's modulus E_n in the direction n_i is determined in the form

$$1/E_n = n_i n_j S_{ijkl} n_k n_l. \tag{8}$$

Poisson's ratio ν_{mn} in the direction m_i under tension in the direction n_i is

$$\nu_{mn}/E_n = -m_i m_j S_{ijkl} n_k n_l \tag{9}$$

(no summation in terms of n is performed on the left). The shear modulus μ_{nm} between the areas with the normals n_i and m_i is

$$1/(4\mu_{nm}) = n_i m_j S_{ijkl} n_k m_l. \tag{10}$$

The bulk modulus K can be presented in the form $1/K = S_{iikk}$. The strain energy (4) should be a positively determined quadratic form [1, 3, 25].

The classical linear theory of elasticity was developed in the 19th century by A. L. Cauchy, C. L. M. H. Navier, S. D. Poisson, G. Green, A. J. C. Barré de Saint-Venant, and other scientists (the history of the elasticity theory is described in [3, 26–31]).

In 1660, R. Hooke discovered the law of proportionality of stresses and strains in the simplest form. Cherepanov [32] said that L. Euler was the first to formulate the law of elasticity in the form $\sigma = E\varepsilon$ [he denoted the constant E by the first letter of his name (Euler)].

In 1821, C. L. M. H. Navier started constructing the elasticity theory. In 1822, A. L. Cauchy introduced the notion of a stress state (as it is understood today) at a point defined by six components σ_{ij} and derived equations of motion and equilibrium. The Cauchy equations are currently accepted for isotropic materials. In 1828, A. L. Cauchy obtained Hooke's law with 21 constants; under certain assumptions, however, the number of constants is reduced to 15 and even to 1 constant for an isotropic material. Equations similar to the Navier and Cauchy equations were also derived by S. D. Poisson (1828). There was a long discussion in papers dealing with the elasticity theory (see, e.g., [3, 26]) on the so-called multi-constant and rare-constant theories, i.e., on the number of independent constants in the generalized Hooke's law (21 or 15 constants for arbitrary anisotropy and, correspondingly, 2 or 1 constant for an isotropic material). Experimental data, in particular, W. Voigt's experiments aimed at studying elastic properties of crystals did not confirm that the six Cauchy conditions [3] $E_{iklj} - E_{ilkj} = 0$ were satisfied, i.e., that the number of independent elasticity moduli was 15. These conditions are approximately satisfied for beryl and rock salt. Only in the beginning of the 20th century, after the papers of M. Born on the crystal lattice theory were published [3, 33], it was finally recognized that Hooke's law in the general case contains 21 constants.

A. L. Cauchy (1830) and G. Green (1839) studied propagation of plane waves in an elastic medium and derived equations for the propagation velocity as a function of the direction of the normal to the wave front [3]. In the general case, the wave surface [25] consists of three surfaces; for an isotropic medium, all three surfaces are spheres, and two of them coincide with each other [3].

The activities of G. Green (1837) were of much importance for the development of the fundamentals of the elasticity theory. G. Green assumed the existence of the specific potential energy, with the stresses being determined by the partial derivatives of this energy with respect to strains. Based on this fact, G. Green derived the equations of the elasticity theory with 21 constants (for the isotropic case, with 2 constants).

A. E. H. Love wrote in [3] that the fact that the resistances to bulk compression and shear are two basic forms of elastic resistance of isotropic solids was first noted by Stokes. Actually, this means the existence of two eigenstates [34, 35] of isotropic materials. The difference between two types of elastic resistance was noted by J.-V. Poncelet in 1839 [3]. The concepts of eigenmoduli of elasticity and eigenstates (under different names) were introduced by W. Thomson (Lord Kelvin) [29, 34, 36] in the middle of the 19th century, but his results were forgotten for a long time. It was only in the last decades that the eigenmoduli and eigenstates were used by a number of researchers, including L. M. Minkevich [37–39], J. Rychlewski [34, 40], A. I. Chanyshev [41–43], N. I. Ostrosablin [35, 44–46], and others (see [47–52]).

The elasticity properties of materials are determined by the fourth-rank tensor of the elasticity moduli E_{ijkl}. At the end of the 19th century to the beginning of the 20th century, F. E. Neumann and W. Voigt described the elastic properties of crystals using the property of symmetry [3, 53]. Depending on the type of symmetry, crystals are divided into seven classes: 1) triclinic; 2) monoclinic; 3) rhombic; 4) tetragonal; 5) trigonal; 6) hexagonal; 7) cubic. Most course manuals in crystallophysics and theory of elasticity of anisotropic materials are based on these results. The following types of symmetry are distinguished in the elasticity theory: isotropy, transverse isotropy, orthotropy (three planes of symmetry) and cubic symmetry.

In the 1920s–1930s, P. Bechterew derived various relations and inequalities for elasticity moduli; he also posed and studied the problem of determining the closest boundaries of the elasticity moduli and compliance coefficients, which provided the positive definiteness of the specific strain energy [6–16]. The importance of this problem was noted by V. V. Novozhilov [1, 17] and K. F. Chernykh [17, 18]. The problem of the closest boundaries for elasticity characteristics was solved in [55, 56].

P. Bechterew proposed a classification of anisotropic materials in terms of their properties approaching the properties of liquids or solids: hygromorphic, orthomorphic, plagiomorphic, and scleromorphic materials. Depending on Poisson's ratio ν, P. Bechterew divided the range of stable isotropic states into two intervals: $0 < \nu < 1/2$ and $-1 < \nu < 0$. Materials corresponding to the first and second intervals were called chorostable and achorostable. One-sided extension of the sample is accompanied by transverse compression in the first case and by transverse extension in the second case. The Poisson's ratio $\nu = 1/2$ corresponds to materials of an ideal fluid type (with a large bulk modulus and a small shear modulus). The Poisson's ratio $\nu = -1$ refers to materials of an ideal solid type (with an extremely large shear modulus and a small bulk modulus).

Following P. Bechterew's papers, we introduce a six-dimensional space of vectors with standard scalar multiplication:

$$\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6), \qquad \boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6),$$

$$\begin{gathered} \sigma_1 = \sigma_{11}, \quad \sigma_2 = \sigma_{22}, \quad \sigma_3 = \sigma_{33}, \quad \sigma_4 = \sqrt{2}\,\sigma_{23}, \quad \sigma_5 = \sqrt{2}\,\sigma_{13}, \quad \sigma_6 = \sqrt{2}\,\sigma_{12}, \\ \varepsilon_1 = \varepsilon_{11}, \quad \varepsilon_2 = \varepsilon_{22}, \quad \varepsilon_3 = \varepsilon_{33}, \quad \varepsilon_4 = \sqrt{2}\,\varepsilon_{23}, \quad \varepsilon_5 = \sqrt{2}\,\varepsilon_{13}, \quad \varepsilon_6 = \sqrt{2}\,\varepsilon_{12}. \end{gathered} \tag{11}$$

Hooke's law (2) acquires the form

$$\sigma_i = A_{ij}\varepsilon_j, \qquad \varepsilon_i = B_{ij}\sigma_j, \qquad i, j = \overline{1, 6}. \tag{12}$$

A symmetric matrix with the components A_{ij} is defined in the form

$$[A_{ij}] = \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & \sqrt{2}\,E_{1123} & \sqrt{2}\,E_{1113} & \sqrt{2}\,E_{1112} \\ E_{2211} & E_{2222} & E_{2233} & \sqrt{2}\,E_{2223} & \sqrt{2}\,E_{2213} & \sqrt{2}\,E_{2212} \\ E_{3311} & E_{3322} & E_{3333} & \sqrt{2}\,E_{3323} & \sqrt{2}\,E_{3313} & \sqrt{2}\,E_{3312} \\ \sqrt{2}\,E_{2311} & \sqrt{2}\,E_{2322} & \sqrt{2}\,E_{2333} & 2E_{2323} & 2E_{2313} & 2E_{2312} \\ \sqrt{2}\,E_{1311} & \sqrt{2}\,E_{1322} & \sqrt{2}\,E_{1333} & 2E_{1323} & 2E_{1313} & 2E_{1312} \\ \sqrt{2}\,E_{1211} & \sqrt{2}\,E_{1222} & \sqrt{2}\,E_{1233} & 2E_{1223} & 2E_{1213} & 2E_{1212} \end{bmatrix}.$$

Similarly, the matrix B_{ij} is defined via the tensor S_{ijkl}, and the equality $A_{ij}B_{jk} = \delta_{ik}$ $(i,j,k=\overline{1,6})$ holds. The advantage of notation (11) was demonstrated in [25, 54].

The technical compliance coefficients $B_{ij} = \nu_{ij}/E_j = \nu_{ji}/E_i$ (no summation is performed with respect to i, j) were also used by Ya. I. Sekerzh-Zen'kovich [57], N. G. Chentsov [58], and A. L. Rabinovich [59]. Direction surfaces, direction curves, and anisotropy diagrams, which show the changes in Young's moduli, Poisson's ratios, and shear moduli, depending on the direction in which these quantities are calculated, were described in [59–61].

It follows from Eq. (5) that there are only 21 independent components E_{ijkl}. After the orthogonal transformation of the coordinate system

$$\hat{x}_j = \alpha_{ij}x_i, \qquad \alpha_{ij}\alpha_{ik} = \delta_{jk} \tag{13}$$

we obtain

$$\hat{\varepsilon}_{kl} = \alpha_{ik}\alpha_{jl}\varepsilon_{ij}, \qquad \hat{E}_{pqrs} = \alpha_{ip}\alpha_{jq}E_{ijkl}\alpha_{kr}\alpha_{ls}.$$

In Eqs. (11), the last relations can be written as

$$\hat{\varepsilon}_i = l_{ki}\varepsilon_k, \qquad \hat{A}_{ij} = l_{si}A_{sk}l_{kj}.$$

The orthogonal matrix l_{ij} has the following form [12]:

$$[l_{ij}] = \begin{bmatrix} \alpha_{11}^2 & \alpha_{12}^2 & \alpha_{13}^2 & \sqrt{2}\,\alpha_{12}\alpha_{13} & \sqrt{2}\,\alpha_{11}\alpha_{13} & \sqrt{2}\,\alpha_{11}\alpha_{12} \\ \alpha_{21}^2 & \alpha_{22}^2 & \alpha_{23}^2 & \sqrt{2}\,\alpha_{22}\alpha_{23} & \sqrt{2}\,\alpha_{21}\alpha_{23} & \sqrt{2}\,\alpha_{21}\alpha_{22} \\ \alpha_{31}^2 & \alpha_{32}^2 & \alpha_{33}^2 & \sqrt{2}\,\alpha_{32}\alpha_{33} & \sqrt{2}\,\alpha_{31}\alpha_{33} & \sqrt{2}\,\alpha_{31}\alpha_{32} \\ \sqrt{2}\,\alpha_{21}\alpha_{31} & \sqrt{2}\,\alpha_{22}\alpha_{32} & \sqrt{2}\,\alpha_{23}\alpha_{33} & \alpha_{22}\alpha_{33}+\alpha_{23}\alpha_{32} & \alpha_{21}\alpha_{33}+\alpha_{23}\alpha_{31} & \alpha_{21}\alpha_{32}+\alpha_{22}\alpha_{31} \\ \sqrt{2}\,\alpha_{11}\alpha_{31} & \sqrt{2}\,\alpha_{12}\alpha_{32} & \sqrt{2}\,\alpha_{13}\alpha_{33} & \alpha_{12}\alpha_{33}+\alpha_{13}\alpha_{32} & \alpha_{11}\alpha_{33}+\alpha_{13}\alpha_{31} & \alpha_{11}\alpha_{32}+\alpha_{12}\alpha_{31} \\ \sqrt{2}\,\alpha_{11}\alpha_{21} & \sqrt{2}\,\alpha_{12}\alpha_{22} & \sqrt{2}\,\alpha_{13}\alpha_{23} & \alpha_{12}\alpha_{23}+\alpha_{13}\alpha_{22} & \alpha_{11}\alpha_{23}+\alpha_{13}\alpha_{21} & \alpha_{11}\alpha_{22}+\alpha_{12}\alpha_{21} \end{bmatrix}.$$

By choosing three free parameters α_{ij} determining the position of the coordinate system (13), the number of independent components E_{ijkl} can be reduced from 21 to 18 [1, 62]. For various cases of symmetry in the structure of anisotropic materials, the number of independent constants E_{ijkl} becomes even smaller [1–3, 25]. Detailed derivation and recording of the matrix of the elasticity moduli A_{ij} with the minimum number of independent constants for all types of crystallographic symmetry can be found in F. I. Fedorov's monograph [25]. The issues of symmetry of the tensor E_{ijkl} with respect to crystallographic groups were also considered in [63–79].

V. V. Novozhilov proposed the convolution $E_{ijkk} = K_{ij}$ and called the principal axes of this tensor — the principal axes of anisotropy [1]. Under all-sided strains ($\varepsilon_{kl} = \varepsilon\delta_{kl}$), the stresses are identical: $\sigma_{ij} = E_{ijkl}\varepsilon\delta_{kl} = E_{ijkk}\varepsilon = K_{ij}\varepsilon$. K. F. Chernykh introduced a special basis [18], which somewhat simplifies the recording of Hooke's law. This basis turned to be an eigenbasis only for isotropic materials and materials with a cubic symmetry and is not an eigenbasis for materials with other types of crystallographic symmetry [45]. A trigonometric presentation of elasticity constants, which provides positive definiteness of the strain energy, was given in [17, 18, 20–22].

Possible forms of the functional relation between two symmetric tensors of the second rank were studied in [80–87]. The decomposition of the deviator into pure shear components was given in [88–90].

Particular constraints on the elasticity constants for some particular materials were considered in [91–95]. The elasticity constants for various substances and crystals and numerous references on this issue can be found in [96–100].

The notion of elasticity eigenmoduli and eigenstates of elastic materials is extremely important for understanding the mathematical and mechanical structure of the tensor E_{ijkl}. The eigenmoduli and eigenstates for isotropic materials have been known since the period of creative activities of G. G. Stokes (see the information given above and [3]). The eigenmoduli and eigenstates were used in the last decades in [34, 35, 37–52, 101–107].

The essence of this approach consists in the following. The eigenmodulus (eigenvalue) Λ and the eigentensor (eigenstate of the elastic material) t_{ij} are determined from the linear system

$$E_{ijkl}t_{kl} = \Lambda t_{ij}, \qquad i,j,k,l = 1,2,3. \tag{14}$$

These exist six eigenmoduli Λ_p $(p=\overline{1,6}\,)$, which correspond to six stress eigentensors $t_{ij}^{(p)}$ satisfying the orthogonality conditions $t_{ij}^{(p)}t_{ij}^{(q)} = \delta_{pq}$. The tensors $t_{ij}^{(p)}$ form an orthonormalized basis induced by the tensor E_{ijkl} in the space of symmetric tensors of the second rank. All symmetric tensors of the second rank, in particular, stress and strain tensors, are decomposed with respect to this basis:

$$\sigma_{ij} = k_1^{(\sigma)} t_{ij}^{(1)} + k_2^{(\sigma)} t_{ij}^{(2)} + \ldots + k_6^{(\sigma)} t_{ij}^{(6)}, \qquad \varepsilon_{ij} = k_1^{(\varepsilon)} t_{ij}^{(1)} + k_2^{(\varepsilon)} t_{ij}^{(2)} + \ldots + k_6^{(\varepsilon)} t_{ij}^{(6)}$$

$$(k_p^{(\sigma)} = \sigma_{ij} t_{ij}^{(p)}, \qquad k_p^{(\varepsilon)} = \varepsilon_{ij} t_{ij}^{(p)}, \qquad p = \overline{1,6}\,).$$

Hooke's relations (2) are equivalent to the equalities

$$k_1^{(\sigma)} = \Lambda_1 k_1^{(\varepsilon)}, \qquad k_2^{(\sigma)} = \Lambda_2 k_2^{(\varepsilon)}, \qquad k_3^{(\sigma)} = \Lambda_3 k_3^{(\varepsilon)},$$

$$k_4^{(\sigma)} = \Lambda_4 k_2^{(\varepsilon)}, \qquad k_5^{(\sigma)} = \Lambda_5 k_2^{(\varepsilon)}, \qquad k_6^{(\sigma)} = \Lambda_6 k_6^{(\varepsilon)}.$$

The quadratic form (4) is reduced to a diagonal form

$$2\Phi = \sum_{p=1}^{6} \Lambda_p (k_p^{(\varepsilon)})^2 = \sum_{p=1}^{6} \frac{(k_p^{(\sigma)})^2}{\Lambda_p},$$

and the condition of its positive definiteness is the positive values of all Λ_p $(p = \overline{1,6}\,)$. The tensor E_{ijkl} can be presented as

$$E_{ijkl} = \sum_{p=1}^{6} \Lambda_p t_{ij}^{(p)} t_{kl}^{(p)}. \tag{15}$$

With allowance for notations (11) and (12), system (14) acquires the form

$$A_{ij} t_j = \Lambda t_i, \qquad i, j = \overline{1,6}. \tag{16}$$

The linear system (16) has a non-zero solution if the eigenmoduli Λ satisfy the equation $|A_{ij} - \Lambda\delta_{ij}| = 0$. By decomposing this determinant, we obtain an equation of the sixths power with respect to Λ

$$\Lambda^6 - I_1\Lambda^5 + I_2\Lambda^4 - I_3\Lambda^3 + I_4\Lambda^2 - I_5\Lambda + I_6 = 0, \tag{17}$$

where the coefficients I_k $(k = \overline{1,6}\,)$ are invariants of the elasticity moduli tensor E_{ijkl} [46]. If the non-zero solutions of system (16) corresponding to the roots of Eq. (17) are orthonormalized as

$$t_i^{(p)} t_i^{(q)} = \delta_{pq}, \qquad i, p, q = \overline{1,6}, \tag{18}$$

then Hooke's law (12) is written in the form

$$\sigma_i t_i^{(1)} = \Lambda_1 \varepsilon_j t_j^{(1)}, \qquad \sigma_i t_i^{(2)} = \Lambda_2 \varepsilon_j t_j^{(2)}, \qquad \sigma_i t_i^{(3)} = \Lambda_3 \varepsilon_j t_j^{(3)},$$

$$\sigma_i t_i^{(4)} = \Lambda_4 \varepsilon_j t_j^{(4)}, \qquad \sigma_i t_i^{(5)} = \Lambda_5 \varepsilon_j t_j^{(5)}, \qquad \sigma_i t_i^{(6)} = \Lambda_6 \varepsilon_j t_j^{(6)}.$$

The elasticity moduli and the compliance coefficients are defined by 6 eigenmoduli $\Lambda_k > 0$ $(k = \overline{1,6}\,)$ and 15 parameters $t_i^{(p)}$, which remain free after the conditions of orthonormalization (18) are satisfied. Three parameters $t_i^{(p)}$ are determined by the choice of the coordinate system, and the remaining parameters are characteristics of the anisotropic material.

J. Rychlewski [34, 40] introduced the term "elastic eigenstate," proposed a classification of anisotropic materials on the basis of the structural formula (15) for E_{ijkl}, and derived explicit formulas for the bulk modulus, Young's moduli, Poisson's ratios, and shear moduli expressed via eigenmoduli and eigenstates. There are some later publications on this topic [35, 41–46, 107]; there are also some earlier works, both domestic [37–39, 103, 105, 106] and foreign [101, 102, 104], where the eigenmoduli and eigenstates for the elasticity moduli tensor E_{ijkl} were considered. Some foreign publications with repeated results obtained previously continue to appear (see, e.g., [47–52, 108]).

Eigenvalues and eigentensors for the tensor E_{ijkl}^* [see Eq. (7)] possessing symmetry of the form (5) were introduced in [23, 24]. Eigenvalues and eigenvectors were found for matrices of the coefficients E_{ijkl}^* of materials with crystallographic symmetry. Depending on the number of different eigenvalues and their multiplicities, the equations of motion (6) are divided into 32 classes. A classification of anisotropic materials slightly different from

J. Rychlewski's classification was proposed in [35, 44–46]. The eigenstates $t_i^{(p)}$ in [35, 44] were constructed in the general form as functions of 15 arbitrary parameters. It was demonstrated that all orthonormalized eigenstates $t_i^{(p)}$ are obtained by means of orthogonalization and normalization of an arbitrary triangular matrix.

For a material traditionally called isotropic, we obtain

$$\Lambda_1 = 3\lambda + 2\mu, \qquad \Lambda_2 = \Lambda_3 = \Lambda_4 = \Lambda_5 = \Lambda_6 = 2\mu, \qquad E_{ijkl} = \lambda\delta_{ij}\delta_{kl} + 2\mu\delta_{ijkl},$$

where λ and μ are the Lamé constants [2, 4], and the eigenstates have the form

$$t^{(1)} = (1/\sqrt{3},\, 1/\sqrt{3},\, 1/\sqrt{3},\, 0,\, 0,\, 0), \qquad t^{(2)} = (1/\sqrt{6},\, 1/\sqrt{6},\, -2/\sqrt{6},\, 0,\, 0,\, 0),$$
$$t^{(3)} = (1/\sqrt{2},\, -1/\sqrt{2},\, 0,\, 0,\, 0,\, 0), \qquad t^{(4)} = (0,\, 0,\, 0,\, 1,\, 0,\, 0), \tag{19}$$
$$t^{(5)} = (0,\, 0,\, 0,\, 0,\, 1,\, 0), \qquad t^{(6)} = (0,\, 0,\, 0,\, 0,\, 0,\, 1).$$

The first eigenstate in $t^{(1)}$ is a spherical tensor, and the remaining eigenstates are deviators.

Anisotropic materials (crystals) are usually classified on the basis of the properties of symmetry with respect to orthogonal transformations of the coordinate system [2, 3, 25]. Anisotropic materials can also be classified depending on the number of different eigenmoduli Λ_p and their multiplicities [34, 35, 40, 44, 46]. Based on the number of different eigenmoduli Λ_p, all anisotropic materials are also divided into groups (classes), which, in turn, are divided into subclasses, depending on the multiplicity of the eigenmoduli. Thus, 32 classes of anisotropic materials are obtained. A more detailed classification of anisotropic materials should be performed on the basis of the form of the eigentensors $t_i^{(p)}$.

Many monographs on the elasticity theory [3, 109, 110] argue that no materials with negative Poisson's ratios were found in experiments. At the moment, however, some composite materials with negative Poisson's ratios have been developed [111–120]. Savrasov [121] intended to prove that no materials with negative Poisson's ratios can exist; he introduced a certain artificial decomposition of the stress and strain states into components and unduly criticized the division of the stress and strain tensors into spherical and deviatoric components, which is conventionally used for isotropic materials.

Ostrosablin [45] found the elasticity eigenmoduli and eigenstates for materials with all types of crystallographic symmetry (see also [37]). In terms of their symmetry properties, these materials (crystals) are subdivided into seven classes and an isotropic medium [25]. For materials with cubic symmetry, the eigenstates $t_i^{(p)}$ are also defined by Eqs. (19). In systems with triclinic symmetry, the matrix A_{ij} has the general form. In terms of symmetry, materials with triclinic symmetry do not differ from each other, but they may be qualitatively different, depending on their eigenmoduli and eigenstates [34, 40, 44, 46]. Chernykh [18] proposed to use basis (19) for materials of all types of symmetry (see also [17, 20–22]).

There are some papers where various presentations of the tensors E_{ijkl} and S_{ijkl}, which differ from the presentations via elastic eigenstates, are given [69, 122–137], constraints ensuring positive definiteness of the specific strain energy are studied [138, 139], and the tensor E^*_{ijkl} [see Eq. (7)] [140–143] and the invariants of the tensor E_{ijkl} [62, 144–157] are considered.

In [55, 56], the elasticity constants are presented as

$$A_{ij} = d_1c_{i1}c_{j1} + d_2c_{i2}c_{j2} + d_3c_{i3}c_{j3} + d_4c_{i4}c_{j4} + d_5c_{i5}c_{j5} + d_6c_{i6}c_{j6}, \tag{20}$$

where

$$c_{ip} = 0 \quad (p > i), \qquad c_{11} = \ldots = c_{66} = 1;$$
$$d_1 > 0, \qquad d_2 > 0, \qquad d_3 > 0, \qquad d_4 > 0, \qquad d_5 > 0, \qquad d_6 > 0. \tag{21}$$

Conditions (21) are necessary and sufficient for positive definiteness of the matrices A_{ij} and B_{ij}. Defining 6 positive numbers d_k and 15 arbitrary parameters c_{ik} $(i > k)$, we apply Eqs. (20) to find the ranges of the quantities A_{ij} or B_{ij} for all anisotropic materials whose properties are described by Hooke's law (12).

Relations (8)–(10) yield the following expressions for the elasticity moduli and Poisson's ratios in the coordinate directions:

$$1/E_1 = B_{11} = d_1, \qquad 1/E_2 = B_{22} = d_1c_{21}^2 + d_2, \qquad 1/E_3 = B_{33} = d_1c_{31}^2 + d_2c_{32}^2 + d_3,$$

$$-\nu_{21} = c_{21}, \qquad -\nu_{31} = c_{31},$$

$$-\nu_{12} = \frac{d_1 c_{21}}{d_1 c_{21}^2 + d_2}, \qquad -\nu_{32} = \frac{d_1 c_{31} c_{21} + d_2 c_{32}}{d_1 c_{21}^2 + d_2},$$

$$-\nu_{13} = \frac{d_1 c_{31}}{d_1 c_{31}^2 + d_2 c_{32}^2 + d_3}, \qquad -\nu_{23} = \frac{d_1 c_{31} c_{21} + d_2 c_{32}}{d_1 c_{31}^2 + d_2 c_{32}^2 + d_3}, \tag{22}$$

$$1/(2\mu_{23}) = B_{44} = d_1 c_{41}^2 + d_2 c_{42}^2 + d_3 c_{43}^2 + d_4,$$

$$1/(2\mu_{13}) = B_{55} = d_1 c_{51}^2 + d_2 c_{52}^2 + d_3 c_{53}^2 + d_4 c_{54}^2 + d_5,$$

$$1/(2\mu_{12}) = B_{66} = d_1 c_{61}^2 + d_2 c_{62}^2 + d_3 c_{63}^2 + d_4 c_{64}^2 + d_5 c_{65}^2 + d_6.$$

In Eqs. (22), the quantities B_{ij} are presented in the form (20). By imparting arbitrary values to the parameters $d_k > 0$, c_{ik} $(i > k)$, n_i, and m_i in Eqs. (8)–(10) and (22), we obtain the admissible limits of variation of the corresponding elasticity constant for an arbitrary anisotropic material. Ostrosablin [56] considered the admissible limits of variation of the elasticity constants for materials with all types of crystallographic symmetry. The limits of variation of Poisson's ratios were considered in [95, 120, 121, 158–163].

Annin et al. [164] studied the following problem of identification of anisotropic materials. Components of the tensor E_{ijkl} possessing the properties of symmetry (5) are specified in a Cartesian coordinate system (x_1, x_2, x_3); components of the fourth-rank tensor Z_{ijkl} related by conditions of symmetry similar to equalities (5) are also specified in this coordinate system; it is necessary to determine if there exists a Cartesian coordinate system $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ [see Eq. (13)] such that the following equalities are satisfied:

$$Z_{ijkl} = \hat{E}_{ijkl}, \qquad i, j, k, l = 1, 2, 3.$$

Computational algorithms were developed and implemented for solving this problem. Another approach to the identification problem is based on constructing a full system of polynomial invariants with respect to orthogonal transformations of coordinates [165]. Such a system has not yet been constructed [34, 166]. This issue was considered in [12, 126, 144–148, 150–153, 155–157]. Note that knowing the invariants of the elasticity moduli tensor is important for the theory of the constitutive relations [146] and rational design of layered composite materials [144].

In accordance with irreducible linear presentations of the orthogonal group of transformations (13) [122, 133–137, 167–169], the tensor E_{ijkl} admits a decomposition into constant, deviatoric, and nonor parts in the form

$$E_{ijkl} = \lambda^* \delta_{ij}\delta_{kl} + \mu^*(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + (M_{ij}\delta_{kl} + M_{kl}\delta_{ij}) + (P_{ik}\delta_{lj} + P_{lj}\delta_{ik} + P_{il}\delta_{jk} + P_{jk}\delta_{il}) + N_{ijkl}, \tag{23}$$

where

$$\lambda^* = (2E_{iikk} - E_{ikki})/15, \qquad \mu^* = (3E_{ikki} - E_{iikk})/30,$$

$$M_{ij} = [5(E_{ijkk} - E_{sskk}\delta_{ij}/3) - 4(E_{ikkj} - E_{skks}\delta_{ij}/3)]/7,$$

$$P_{ij} = [-2(E_{ijkk} - E_{sskk}\delta_{ij}/3) + 3(E_{ikkj} - E_{skks}\delta_{ij}/3)]/7.$$

The following equalities are satisfied thereby:

$$M_{ii} = P_{ii} = 0, \qquad N_{ijkk} = 0, \quad i, j, k = 1, 2, 3.$$

The tensor N_{ijkl}, called the nonor, has nine independent components and is symmetric over all pairs of subscripts. Each of the deviators M_{ij} and P_{ij} has five independent components. The constants λ^* and μ^* are also independent. The tensor S_{ijkl} also admits a decomposition of the form (23). Ostrosablin [169] gave all values of the right side of Eq. (23) for all types of crystallographic symmetry and other variants of decompositions of the form (23). Annin [170] proposed a tensor E_{ijkl} of the form

$$E_{ijkl} = a\delta_{ij}\delta_{kl} + b(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + c(h_{ij}H_{kl} + h_{kl}H_{ij} + h_{ik}H_{lj} + h_{lj}H_{ik} + h_{il}H_{jk} + h_{jk}H_{il}),$$

where $h_{ij} = p_i q_j + q_i p_j$, $H_{ij} = p_i p_j - q_i q_j$, p_i and q_j are the components of the orthogonal unit vectors, and a, b, and c are constants. The expression in brackets at the coefficient c is a nonor.

Some general issues associated with anisotropy were considered in [171, 172]. Examples of elastic anisotropic materials possessing unusual properties were given in [167, 168, 173, 174] with the use of linear invariant irreducible decompositions of the elasticity moduli tensor. Explicit formulas for anisotropic materials conducting purely longitudinal and transverse waves with an arbitrary direction of the wave normal were obtained in [175–178]. The existence of such media was noted in [167, 173, 174]. Rychlewski [167] and Ostrosablin [177] gave examples of anisotropic materials in which Young's modulus $1/E_n = n_i n_j S_{ijkl} n_k n_l$ in the direction n_i is independent of n_i and is identical for all directions ($1/E_n = B_{11}$), as in the case of an isotropic material.

The concepts of elastic eigenstates found application in constructing equations of the plasticity theory [41–43, 179–194] and yield criteria [103, 108, 195–202], in studying elastic solids with constraints [104, 203–206], and in the theory of nonlinear anisotropic elasticity [207].

Rychlewski [195] considered details of the decomposition of the specific strain energy $2\Phi = \sigma_{ij} S_{ijkl} \sigma_{kl}$ and the quadratic criteria of the limiting elastic state $\sigma_{ij} H_{ijkl} \sigma_{kl} \leqslant 2k^2$. This approach was also developed in [197–200]. Criteria of strength and fracture of anisotropic media were considered in [202, 208–210].

An attempt was made in [211–213] to use elastic eigenstates for studying the equations of motion of anisotropic elastic solids. The limits of variation of the elasticity constants, extreme values of Young's moduli, shear moduli, and Poisson's ratios were studied in [159–161, 214–219].

Nontraditional materials with negative Poisson's ratios were considered in [111–118]. Attempts were made in some papers [95, 121, 158] to prove that Poisson's ratio cannot be negative. Ostrosablin [24] gave matrices of elasticity moduli for materials with negative Poisson's ratios; the equations of motion for each displacement are independent of each other. Konek et al. [120] systematized the information about materials with negative Poisson's ratios, which are called auxetics.

The notions of the eigenmoduli and eigenstates were used in [220–240], and various aspects of anisotropy in the linear theory of elasticity were studied. Scott [241] proposed elasticity moduli that take into account the strain-induced changes in element areas. The issues of anisotropy of the elasticity properties in a two-dimensional case were considered in [242, 243]. The properties and applications of the fourth-rank tensors of the form (5) were also considered in [244–247].

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